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Comments on effective superpotentials via matrix models

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Abstract: Dijkgraaf and Vafa have conjectured that the effective superpotentials for $\mathcal{N} = 1$ four-dimensional supersymmetric gauge theories can be given by the planar diagrams of matrix models. We examine some special models with cubic and quartic tree level superpotentials for adjoint chiral superfield Φ. We consider the effective superpotentials for the classical vacuum $\Phi = 0$ for $\mathrm{U}(N)$ and $\mathrm{SO}(N)/\,\mathrm{Sp}(N)$ gauge theories. We evaluate the effective superpotentials exactly in terms of the matrix model and in terms of closed string theory on Calabi-Yau geometry with fluxes. As a result we find their perfect agreements.

Keywords: 1/N Expansion, D-branes, String Duality, Matrix Models.

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1. Introduction

Since 't Hooft introduced an idea [1] of the correspondence between the large-N gauge theory with certain string theory, many dualities had been discussed with this idea. One example of these dualities is the relation between Chern-Simons gauge theory and A-type topological strings [2]–[5]. This duality between the topological theories can be embedded in type-IIA/IIB superstring [6]. The discussion for type-IIB superstrings was extended in [7]–[13]. The equivalence of effective superpotential for $\mathcal{N}=1$ gauge theory with that of type-IIB superstring theory on Calabi-Yau manifold with fluxes [14, 15] was proved in [13].

Recently Dijkgraaf and Vafa conjectured the correspondence between $\mathcal{N}=1$ gauge theories with matrix models based on the reduction of B-type topological string theory with B-branes [16]. They proposed the effective superpotentials for a large class of $\mathcal{N}=1$ super Yang-Mills theories are computed from matrix models [16, 17, 18]. In particular the effective superpotentials for the gauge theories are given by the genus zero free energies which are the summations of planar diagrams in matrix models. This perturbative sum of planar diagrams turns out to be a nonperturbative sum over fractional instantons in the $\mathcal{N}=1$ gauge theory. For $\mathcal{N}=1^*$ theory they found the coincidence with those of field theory results. In [20, 21] Dijkgraaf-Vafa's proposal was tested further for a family of deformations of $\mathcal{N}=4$ super Yang-Mills theory involving an arbitrary polynomial superpotential for one of the three adjoint chiral superfields.

It is important to check this proposal for $\mathcal{N}=1$ super Yang-Mills theory which is the deformed $\mathcal{N}=2$ super Yang-Mills theory by the superpotential for adjoint chiral superfields. In particular the models with cubic and quartic superpotentials are quite interesting, because the matrix model corresponding to these potentials have been studied well and the effective superpotential for the gauge theory was explicitly computed up to fourth order in glueball superfields from the dual closed string theory on Calabi-Yau manifold with RR fluxes [7, 11]. Therefore these models are appropriate for the check of the Dijkgraaf-Vafa conjecture for $\mathcal{N}=1$ theories.

The elegant proof of this correspondence for $\mathcal{N}=1$ super Yang-Mills theory is given in [19]. It is also very important to find the effective superpotential directly for this case and check this correspondence explicitly. In this note we explicitly compute the effective superpotentials for $\mathrm{U}(N)$ and $\mathrm{SO}(N)/\mathrm{Sp}(N)$ gauge theories with cubic/quartic tree level superpotentials around the classical vacuum $\Phi=0$ in the context of matrix model and the context of Calabi-Yau manifold with fluxes. The condition $\Phi=0$ means the gauge symmetry remain unbroken. While preparing the manuscript for submission, we recieved [22] which overlaps with $\mathrm{U}(N)$ gauge theory result in our work.

The organization of this paper is as follows: In section 2, we discuss the effective superpotential for the $\mathrm{U}(N)$ gauge theory with cubic tree level superpotential. At first we review the Calabi-Yau manifold which is the dual of the gauge theory and compute the effective superpotential around the classical vacuum $\Phi=0$. Next we review the derivation of genus zero free energy of the hermitean matrix model and compute its derivative with respect to $S=g_sN$. Comparing these results, we can check the Dijkgraaf-Vafa conjecture exactly for this case. In section 3, we discuss the effective superpotential for the $\mathrm{SO}(N)/\mathrm{Sp}(N)$ gauge theory with quartic tree level superpotential as in section 2.

2. $\mathrm{U}(N)$ gauge theory with the cubic superpotential

In this section we discuss the effective superpotential for the $\mathrm{U}(N)$ gauge theory with cubic tree level superpotential around the classical vacuum $\Phi=0$. Since the equivalence of the effective superpotential for the gauge theory with that of type-IIB superstring theory on Calabi-Yau manifold with fluxes [14, 15] was proved in [13], we concentrate on the Calabi-Yau manifold with fluxes. At first we calculate the effective superpotential from the Calabi-Yau manifold with fluxes. In the classical vacuum $\Phi=0$, since there is only one period of the Calabi-Yau manifold, which is identified with glueball superfield, we can obtain the exact result. Next we consider the effective superpotential from matrix model context in terms of the Dijkgraaf-Vafa conjecture. The hermitean matrix model with cubic action is also exactly solvable, therefore we can obtain the genus zero free energy exactly. Then we discuss the Dijkgraaf-Vafa conjecture with these results.

2.1 The analysis for the Calabi-Yau geometry with fluxes

We begin with the review of the effective superpotential from the Calabi-Yau manifold with fluxes [7]. We consider $4d$ $\mathcal{N}=1$, $\mathrm{U}(N)$ super Yang-Mills theory with tree level superpotential for adjoint chiral superfield Φ

$$W_{\text{tree}}=\frac{m}{2}\mathrm{Tr}\Phi^2+\frac{g}{3}\mathrm{Tr}\Phi^3\,. \tag{2.1}$$

Since the eigenvalues of Φ are given by the roots of $W'_{\text{tree}}=0$, the gauge group is broken as

$$\mathrm{U}(N)\to\mathrm{U}(N_1)\times\mathrm{U}(N_2)\,,\qquad N_1+N_2=N\,. \tag{2.2}$$

The dual description of the gauge theory in this vacuum [7] was given by the following deformed Calabi-Yau manifold with RR-flux H^{RR} and NSNS B-field H^{NS},

$$W'_{\text{tree}}{}^2(x) + y^2 + z^2 + w^2 + f_1(x) = 0\,. \tag{2.3}$$

where W_{tree} is the tree level superpotential for the gauge theory and f_1 is a degree one polynomial. In this Calabi-Yau manifold we can form an integral basis of 3-cycles, A_i and B_i, which form a symplectic paring

$$(A_i, B_j) = -(B_j, A_i) = \delta_{ij}\,, \qquad (A_i, A_j) = (B_i, B_j) = 0\,, \quad i,j = 1,2\,. \tag{2.4}$$

The above geometry have N_i units of RR-flux through the A_i cycle, and an NS-flux α through each of the dual noncompact B_i cycles. α is identified with the complex bare gauge coupling of the gauge theory. The periods of the Calabi-Yau manifold is given by the integral of the holomorphic 3-form Ω over these cycles. Since B_i-cycles are noncompact cycle, we need the cut-off Λ_0 for the integral of B_i-cycle

$$\int_{A_i} \Omega \equiv S_i\,, \qquad \int_{B_i}^{\Lambda_0} \Omega \equiv \Pi_i\,. \tag{2.5}$$

The period integrals over A_i cycle are identified with the glueball superfield $S = \frac{1}{32\pi}\mathrm{Tr}W_\alpha W^\alpha$ whose lowest component is the gluino bilinear. These period integrals of the holomorphic 3-form can be written as integrals of the effective one-form ω over projections of the cycles to the x-plane. The one-form ω is given by doing the Ω integral over the S^2 cycle corresponding to the y, z, w coordinates

$$S_i = \frac{1}{2\pi i}\int \omega\,, \qquad \Pi_i = \frac{1}{2\pi i}\int^{\Lambda_0} \omega\,, \qquad \omega = dx\sqrt{W'^2(x) + f_{n-1}(x)}\,. \tag{2.6}$$

We want to consider the special vacuum in which the gauge symmetry is unbroken $\mathrm{U}(N) \to \mathrm{U}(N)$ [1]. In this classical vacuum there is only one period and the x-plane have one brunch-cut. In this case the effective one form is written as

$$dx\sqrt{W'^2(x) + f_1(x)} = dx g(x + \Delta + x_3)\sqrt{(x - 2x_1)(x - 2x_2)}\,. \tag{2.7}$$

where $\Delta = m/g$. Comparing the coefficients of x^3 and x^2, we get the following relations

$$x_3 = x_1 + x_2\,, \qquad (x_1 - x_2)^2 + 2(x_1 + x_2)[\Delta + x_1 + x_2] = 0\,. \tag{2.8}$$

It is also convenient to introduce new variables given by $\bar\sigma \equiv x_1 + x_2, \mu \equiv x_1 - x_2$. Using these variables, we can rewrite the latter relation of (2.8)

$$\mu^2 + 2\bar\sigma[\Delta + \bar\sigma] = 0\,. \tag{2.9}$$

The period integral is explicitly evaluated,

$$\frac{S}{g} = \frac{1}{2\pi i}\int_{2x_1}^{2x_2} dx(x + \Delta + \bar\sigma)\sqrt{(x - 2x_1)(x - 2x_2)} = \frac{\Delta + 2\bar\sigma}{4}\mu^2\,, \tag{2.10}$$

[1] In the dual gauge theory, this case correspond to all of N D5-branes wrap only on the blown-up $\mathbf{S^2}$ at the origin.

Using this result (2.10), we rewrite the constraint (2.9)

$$2\frac{S}{g} + \bar{\sigma}(\Delta + 2\bar{\sigma})(\bar{\sigma} + \Delta) = 0\,. \tag{2.11}$$

$\bar{\sigma}$ is solved by a power series in S around $\bar{\sigma} = 0$,

$$\bar{\sigma} = -\frac{\Delta}{4}\sum_{k=1}^{\infty}\left(\frac{8S}{g\Delta^3}\right)^k \frac{\Gamma\left((3k-1)/2\right)}{\Gamma\left((k+1)/2\right)}. \tag{2.12}$$

The dual period is written as the function of σ which have the constraint (2.11)

$$\begin{aligned}\frac{2\pi i}{g}\Pi &= \int_{2x_2}^{\Lambda_0}(x + \bar{\sigma} + \Delta)\sqrt{(x - 2x_1)(x - 2x_2)}dx \\ &= \frac{1}{g}W(\Lambda_0) - \frac{\bar{\sigma}^2}{6}(2\bar{\sigma} + 3\Delta) + \frac{S}{g}\left[\log\Lambda_0 + \log\Delta - \log(2\bar{\sigma} + \Delta)\right].\end{aligned} \tag{2.13}$$

The effective superpotential generated from fluxes is given [15] by

$$\begin{aligned}W_{\text{eff}} &= \int \Omega \wedge (H^{RR} + \tau H^{NS}) = 2\pi i(N\Pi - \tau S) \qquad (2.14)\\ &= NS\log\left(\frac{S}{\Lambda_0^3}\right) - 2\pi i\tau S - \frac{Ng\bar{\sigma}^2}{6}(2\bar{\sigma} + 3\Delta) + NS\left[\log\Lambda_0 + \log\Delta - \log(2\bar{\sigma} + \Delta)\right]\end{aligned}$$

where we ignored the irrelevant constant $W(\Lambda_0)$. Thus we found the effective superpotential exactly. Substituting the power series result (2.12) to the exact result (2.14), we obtain

$$\begin{aligned}W_{\text{eff}}(S) &= NS\log\left(\frac{S}{\Lambda_0^3}\right) - 2\pi i\tau S + \frac{N}{2}\sum_{k=1}^{\infty}\left(\frac{8g^2}{m^3}\right)^k \frac{S^{k+1}\Gamma(3k/2)}{(k+1)!\,\Gamma(k/2+1)} \\ &= \cdots + N\left[2S^2 + \frac{32}{2}S^3 + \frac{280}{3}S^4 + \cdots\right].\end{aligned} \tag{2.15}$$

The coefficients of first three terms agree with those of the result already known in [7].

2.2 Hermitean matrix model analysis

The matrix model describing the U(N) gauge theory with tree level superpotential (2.1) is discussed in [16]. For this matrix model the partition function is given by

$$Z = \frac{1}{\text{Vol}\,(\text{U}(N))}\int d\Phi \cdot \exp\left(-\frac{1}{g_s}\text{Tr}W_{\text{tree}}(\Phi)\right), \tag{2.16}$$

where Φ is a $N \times N$ hermitean matrix. This partition function is evaluated perturbatively by double line Feynman diagrams with orientation and has a consistent saddle-point approximation of the form

$$Z = \exp\sum_{g\geq 0} g_s^{2g-2}\mathcal{F}_g(S)\,, \tag{2.17}$$

in the 't Hooft limit $N \gg 1$ and $g_s \ll 1$, while keeping finite the 't Hooft coupling

$S = g_s N$ [16]. $\mathcal{F}_g$ is given by the sum of all diagrams with genus g. As discussed in ordinary quantum field theory, the exponentiation of the connected diagrams generates all diagrams, so the free energy contributes only from connected diagrams.

Since the eigenvalues λ_i $(1 \leq i \leq N)$ of Φ become continuous in the large-N limit, let us introduce a function $\lambda(x)$ such that $\lambda_i = \lambda(i/N)$. Then the genus zero free energy of the model may be given by the integral,

$$\mathcal{F}_0 = -\frac{g_s^2 N^2}{g_s} \int_0^1 dx \left(\frac{m}{2N} \lambda^2(x) + \frac{g}{3N} \lambda^3(x) \right) + g_s^2 N^2 \int_0^1 \int_0^1 dx dy \log |\lambda(x) - \lambda(y)| \,. \tag{2.18}$$

The second term comes from the Vandermonde determinant corresponding to the jacobian. The details of the computation of this genus zero free energy have been discussed in [23].

First of all we introduce the density of eigenvalues $\rho(\lambda)$ defined by

$$2\frac{dx}{d\lambda} = \rho(\lambda)\,, \qquad \int_{2a}^{2b} d\lambda\, \rho(\lambda) = 2\,. \tag{2.19}$$

Outside the interval $(2a, 2b)$ the value is zero. Using this quantity, we obtain the equation of motion in the large-N limit

$$\frac{m}{g_s N}\lambda + \frac{g}{g_s N}\lambda^2 = ⨍_{2a}^{2b} d\mu \frac{\rho(\mu)}{\lambda - \mu} \tag{2.20}$$

where the bar of the integral means the principal part. The solution is obtained by introducing the resolvent,

$$\omega(\lambda) = \int_{2a}^{2b} d\mu \frac{\rho(\mu)}{\lambda - \mu} \tag{2.21}$$

where complex λ is defined outside of the real interval $(2a, 2b)$. It behaves as $2/\lambda$ when $|\lambda|$ goes to infinity. The factor 2 comes from the definition (2.19). When λ approaches the interval $(2a, 2b)$, it behaves as

$$\omega(\lambda \pm i0) = \frac{m}{g_s N}\lambda + \frac{g}{g_s N}\lambda^2 \mp i\pi\rho(\lambda)\,. \tag{2.22}$$

There is a unique function, which satisfies these requirements,

$$\omega(\lambda) = \frac{m}{g_s N}\lambda + \frac{g}{g_s N}\lambda^2 - \left(\frac{m}{g_s N} + \frac{g}{g_s N}(a+b) + \frac{g}{g_s N}\lambda \right) \sqrt{(\lambda - 2a)(\lambda - 2b)}\,, \tag{2.23}$$

with the following constrains,

$$\begin{aligned} 0 &= 2(a+b)\left[m + g(a+b)\right] + g(b-a)^2 \\ 4g_s N &= (b-a)^2 \left[m + 2g(a+b)\right]. \end{aligned} \tag{2.24}$$

Since only the singular part contributes to the density of eigenvalues, we obtain the density of eigenvalues,

$$\begin{aligned} \rho(\lambda) &= \frac{1}{2\pi i}\left(\omega(\lambda + i0) - \omega(\lambda - i0)\right) \\ &= \frac{1}{\pi}\left(\frac{m}{g_s N} + \frac{g}{g_s N}(a+b) + \frac{g}{g_s N}\lambda^2 \right) \sqrt{(\lambda - 2a)(2b - \lambda)}\,. \end{aligned} \tag{2.25}$$

It is convenient to introduce the single parameter $\sigma = \frac{g}{m}(a+b)$. Using this parameter, we may rewrite the constraints (2.24) as

$$\frac{2g^2 N g_s}{m^3} + \sigma(1+\sigma)(1+2\sigma) = 0\,. \tag{2.26}$$

The expansion of σ as a power series in g is given by

$$\sigma = -\frac{1}{4}\sum_{k=1}^{\infty}\left(\frac{8g^2 N g_s}{m^3}\right)^k \frac{\Gamma((3k-1)/2)}{\Gamma((k+1)/2)}\,. \tag{2.27}$$

We rewrite the genus zero free energy in terms of these quantities as

$$\mathcal{F}_0 = -\frac{g_s^2 N^2}{2g_s}\int_{2a}^{2b} d\lambda\rho(\lambda)\left(\frac{m}{2N}\lambda^2(x) + \frac{g}{3N}\lambda^3(x)\right) + \frac{g_s^2 N^2}{4}\int_{2a}^{2b}\int_{2a}^{2b} d\lambda d\mu \rho(\lambda)\rho(\mu)\log|\lambda-\mu|\,. \tag{2.28}$$

With the explicit expression (2.25), we can integrate out the λ.

$$\mathcal{F}_0 = \frac{g_s^2 N^2}{3}\frac{\sigma(3\sigma^2+6\sigma+2)}{(1+\sigma)(1+2\sigma)^2} - \frac{g_s^2 N^2}{2}\log(1+2\sigma)\,. \tag{2.29}$$

This is the exact result of genus zero free energy. Using the relation (2.26), (2.27) we get the genus zero free energy as a power series in $g_s N$,

$$\mathcal{F}_0 = \frac{g_s^2 N^2}{2}\sum_{k=1}^{\infty}(\frac{8g^2 g_s N}{m^3})^k \frac{\Gamma(3k/2)}{(k+2)!\,\Gamma(k/2+1)}\,. \tag{2.30}$$

For the later discussion for the examination of Dijkgraaf-Vafa conjecture, we calculate the derivative of the genus zero free energy with respect to 't Hooft coupling $S \equiv g_s N$.

$$\frac{\partial \mathcal{F}_0}{\partial S} = \frac{2S}{3}\frac{\sigma(3\sigma^2+6\sigma+2)}{(1+\sigma)^2(1+2\sigma)^2} - S\log(1+2\sigma) + S^2\left(\frac{\partial\sigma}{\partial S}\right)\frac{3+14\sigma+20\sigma^2+12\sigma^3+4\sigma^4}{(1+\sigma)^2(1+2\sigma)^3}\,. \tag{2.31}$$

Using the derivative of constraint (2.26) with respect to S

$$\frac{\partial\sigma}{\partial S} = -\frac{2g^2}{m^3}\frac{1}{1+6\sigma+6\sigma^2}\,, \tag{2.32}$$

we obtain

$$\frac{\partial \mathcal{F}_0}{\partial S} = -\frac{m^3}{6g^2}\sigma^2(3+2\sigma) - S\log(1+2\sigma)\,. \tag{2.33}$$

Since $\bar{\sigma} = \Delta\sigma$, this result agree with the fractional instanton contribution part of the dual period integral Π (2.13) in the previous subsection.

2.3 The check of Dijkgraaf-Vafa conjecture for $\mathrm{U}(N)$ case

In [15] the effective superpotential is given by the period S of Calabi-Yau manifold (2.3)

$$W_{\text{eff}}(S) = NS\log\left(\frac{S}{\Lambda_0^3}\right) - 2\pi i\tau S + N\frac{\partial F_{\text{pert}}(S)}{\partial S}\,, \qquad F_{\text{pert}}(S) = \sum_i c_i S^i \tag{2.34}$$

where $F_{\rm pert}(S)$ is the prepotential for the special geometry of (2.3) and expanded perturbatively in terms of S. This superpotential corresponds to that of gauge theory in the confining phase by identifying the periods S with the glueball superfield $S = \frac{1}{32\pi^2}{\rm Tr}W_\alpha W^\alpha$. Under this identification, the first two terms correspond to the Veneziano-Yankielowicz form [24] and τ is the bare complex coupling of the gauge theory.

In [18] it is conjectured that $F_{\rm pert}(S)$ equals to the free energy $\mathcal{F}_0(g_s)$ of the matrix model under the identification between the glueball superfield S with the 't Hooft coupling $g_s N$.

$$\begin{aligned} W_{\rm eff}(S) &= NS\log\left(\frac{S}{\Lambda_0^3}\right) - 2\pi i\tau S + N\frac{\partial \mathcal{F}_0}{\partial S} \\ &= NS\log\left(\frac{S}{\Lambda_0^3}\right) - 2\pi i\tau S - N\left[\frac{m^3}{6g^2}\sigma^2(3+2\sigma) + S\log(1+2\sigma)\right] \end{aligned} \tag{2.35}$$

where the first two terms come from the volume factor in (2.16). The discussion of this volume factor is discussed in [4, 16, 25]. This effective superpotential agrees exactly with the result (2.14) from the Calabi-Yau manifold with fluxes. Furthermore, using the result (2.30) we have easily got the exact effective superpotential for the gauge theory with the tree level superpotential of (2.1)

$$\begin{aligned} W_{\rm eff}(S) &= NS\log\left(\frac{S}{\Lambda_0^3}\right) - 2\pi i\tau S + \frac{N}{2}\sum_{k=1}^{\infty}\left(\frac{8g^2}{m^3}\right)^k \frac{S^{k+1}\Gamma(3k/2)}{(k+1)!\,\Gamma(k/2+1)} \\ &= \cdots + N\left[2S^2 + \frac{32}{2}S^3 + \frac{280}{3}S^4 + \cdots\right]. \end{aligned} \tag{2.36}$$

This is the exact effective superpotential summed up all instanton corrections and agrees perfectly with (2.15). Thus we find that $F_{\rm pert}(S) = \mathcal{F}_0(g_s)$ holds exactly. Therefore Dijkgraaf-Vafa's conjecture is checked for the special vacuum $\Phi = 0$.

3. SO(N)/ Sp(N) gauge theories with quartic superpotential

In this section we discuss the effective superpotential for SO(N)/ Sp(N) gauge theories with quartic tree level superpotential around the classical vacuum $\Phi = 0$. We proceed the discussion in the same way as section 3. But there are some differences. The geometry dual to the gauge theory has Z_2 identification. The matrix models corresponding to the SO(N)/ Sp(N) gauge theories are not hermitean but real symmetric and self-dual quaternionic matrix model with even function tree level superpotential respectively.

3.1 The analysis for the Calabi-Yau geometry with fluxes

We begin with somewhat general discussion for the SO / Sp gauge theories for the later discussion of matrix model. In order to realize SO(N)/ Sp(N) gauge theories, we need introduce orientifold-plane. When an orientifold-plane is introduced, the Calabi-Yau manifold must be invariant under the complex conjugation [3, 10]

$$(x, y, z, w) \to (\bar{x}, \bar{y}, \bar{z}, \bar{w})\,. \tag{3.1}$$

Then we can only realize an SO / Sp gauge theories with tree level superpotential of even function. The large-N dual of this theory is found via geometric transition [10, 11]. Through the geometric transition, the dual geometry for the $\mathrm{SO}(N)/\mathrm{Sp}(N)$ gauge theories is given by

$$W'^{\,2}_{\mathrm{tree}}(x) + y^2 + z^2 + w^2 + f_{2n-2}(x) = 0\,, \qquad W_{\mathrm{tree}} = \sum_{k=1}^{n+1} \frac{g_{2k}}{2k} x^{2k} \tag{3.2}$$

where W_{tree} is the tree level superpotential for the gauge theory and f_{2n-2} is a degree $2n-2$ polynomial.

In the following we concentrate on the case with the quartic tree level superpotential,

$$W_{\mathrm{tree}} = \frac{m}{2}\mathrm{Tr}\Phi^2 + \frac{g}{4}\mathrm{Tr}\Phi^4\,. \tag{3.3}$$

As the case of $\mathrm{U}(N)$ gauge theory, we can define an integral basis of 3-cycles which form a symplectic paring and the periods of the Calabi-Yau manifold. In this case since we also consider the special classical vacuum $\Phi = 0$, there is only one period and the dual period and the x-plane also have one brunch-cut. We write the effective one form as follows

$$2dx\sqrt{W'^2 + f_2} = 2dxg(x^2 + \Delta + x_3)\sqrt{(x-2\mu)(x+2\mu)}\,. \tag{3.4}$$

where $\Delta = m/g$. From the coefficient of x^4 we get the following relations, $x_3 = 2\mu^2$. The period integral is explicitly evaluated,

$$\frac{S}{g} = \frac{1}{\pi i}\int_{-2\mu}^{2\mu} dx(x^2 + \Delta + 2\mu^2)\sqrt{4\mu^4 - x^2} = 6\mu^4 + 2\Delta\mu^2\,. \tag{3.5}$$

In terms of S/g, μ^2 can be solved

$$\mu^2 = \frac{-\Delta + \sqrt{\Delta^2 + 6S/g}}{6}\,.$$

The dual period is evaluated as

$$\begin{aligned} \frac{\pi i}{g}\Pi &= \int_{2\mu}^{\Lambda_0} dx\,\left(x^2 + \Delta + 2\mu^2\right)\sqrt{4\mu^4 - x^2} \\ &= \frac{1}{g}W(\Lambda_0) - \frac{S}{g}\log\Lambda_0 + \frac{S}{2g}\log\frac{S}{2g} - \frac{S}{2g} - \frac{1}{2}\left(\Delta\mu^2 - \frac{S}{2g}\right) + \frac{S}{2g}\log\mu^2\,. \end{aligned} \tag{3.6}$$

The effective superpotential generated from fluxes [10] is given by

$$\begin{aligned} W_{\mathrm{eff}} &= \int \Omega \wedge (H^{RR} + \tau H^{NS}) = 2\pi i\left[(N+2)\,\Pi - \tau S\right] \\ &= (N \mp 2)\,S\log\left(\frac{S}{\Lambda_0^3}\right) - 2\pi i\tau S + \\ &\quad +(N \mp 2)\left[-2S\log\Lambda_0 + S\log\frac{S}{g} + S - \frac{S}{2}\left(\mu^2 - 1\right) + S\log\mu^2\right] \end{aligned} \tag{3.7}$$

where we ignored the irrelevant constant $W(\Lambda_0)$. In the above expression, upper signs are taken for $\mathrm{SO}(N)$ gauge theory and lower signs are taken for $\mathrm{Sp}(N)$ gauge theory. We obtained the effective superpotential for the gauge theory with tree level superpotential (3.3). From the relation (3.5) μ is written as a function of S. Substituting to the effective superpotential, we obtain

$$
\begin{aligned}
W_{\mathrm{eff}}(S) &= \hat{N} S \log\left(\frac{S}{\Lambda_0^3}\right) - 2\pi i \tau S - \hat{N} \sum_{k=1}^{\infty} \left(-\frac{3g}{2m^2}\right)^k \frac{S^{k+1}}{2} \frac{(2k-1)!}{k!(k+1)!} \\
&= \cdots + \hat{N}\left[\frac{3}{2}\left(\frac{g}{4m^2}\right) S^2 - \frac{9}{2}\left(\frac{g^2}{8m^4}\right) S^3 + \frac{45}{2}\left(\frac{g^3}{16m^6}\right) S^4 + \cdots\right], \qquad (3.8)
\end{aligned}
$$

where $\hat{N} = N \mp 2$. The coefficients of the first three terms agree with those of our result in our previous work [11].

3.2 Matrix model analysis

First we discuss somewhat general properties of real symmetric matrix model and self-dual quaternionic matrix model with the potential,

$$W = \sum_{k=0}^{n} u_{2k} \Phi^{2k} . \qquad (3.9)$$

In this case the partition function of the corresponding matrix models is given by

$$Z = \frac{1}{\mathrm{Vol}\,(G)} \int d\Phi \cdot \exp\left(-\frac{1}{g_s} \mathrm{Tr} W(\Phi)\right) \qquad (3.10)$$

where Φ is $N \times N$ real symmetric matrix for $G = \mathrm{SO}(N)$ and $N \times N$ self-dual quaternionic matrix for $G = \mathrm{Sp}(N)$. For these matrix models, the Feynman diagrams become unoriented double line diagrams. So the large-N-limit of this model may be described in terms of a simple Φ^4-theory with single lines in which all non-planar diagrams are omitted.

In this case the analysis which is discussed for hermitean matrix model in [16] is slightly changed. In fact the power of the Vandermonde determinant Δ is not squared and the partition functions become [26, 27]

$$
\begin{aligned}
Z &= \int \prod_{i=1}^{N} d\lambda_i \cdot \Delta(\lambda) \cdot \exp\left(-\frac{1}{g_s} \sum_{i=1}^{N} W(\lambda_i)\right) \quad \text{for } \mathrm{SO}(N)\,, \\
Z &= \int \prod_{i=1}^{N/2} d\lambda_i \cdot \Delta(\lambda)^4 \cdot \exp\left(-\frac{1}{g_s} \sum_{i=1}^{N/2} 2W(\lambda_i)\right) \quad \text{for } \mathrm{Sp}(N) \qquad (3.11)
\end{aligned}
$$

where N is even for $\mathrm{Sp}(N)$.

We proceed the discussion of loop equation as in [16].[2] In the large-N limit the loop equation becomes

$$\omega^2(x) + \frac{2}{\mu}\omega(x) W'(x) + \frac{1}{\mu^2} f(x) = 0 \qquad (3.12)$$

[2]Here we will concentrate on the $\mathrm{SO}(N)$ case. For the $\mathrm{Sp}(N)$ case there is slight change of the coefficient in (3.12), but they can be absorbed in the normalizations of $f(x)$ and $y(x)$.

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where $\mu = g_s N$ is the 't Hooft coupling, ω is the resolvent and $f(x)$ is given by

$$f(x) = \frac{2\mu}{N} \sum_i \frac{W'(x) - W'(\lambda_i)}{x - \lambda_i} \,. \tag{3.13}$$

We have an algebraic loop equation because of the large-N limit. Let us introduce a new variable,

$$y(x) = \mu\, \omega(x) + W'(x) \,. \tag{3.14}$$

Using this variable we can rewrite the loop equation (3.12) as

$$y^2 - W'(x)^2 + f(x) = 0 \,. \tag{3.15}$$

Thus we obtain a (singular) hyperelliptic curve in the (x, y)-space. The remaining analysis is the same as in [16]. In the following, we will evaluate the derivative of the free energy $\mathcal{F}_0(\mu) \equiv g_s^2 \log Z$ in terms of the 't Hooft coupling μ.

Let us remark about the hyperelliptic curve. As in [16, 28] from the relation (3.13) we can obtain the coefficient of $f(x) = \sum_{k=0}^{n-1} 2\mu b_k x^k$ explicitly,

$$b_{2k} = (2k+2)u_{2k+2} + \sum_{s=1}^{n+1} (2k+2s) u_{2k+2s} \frac{\partial \mathcal{F}_0}{\partial u_{2s-2}} \,, \quad b_{2k+1} = 0 \,, \quad k = 1, \ldots, \frac{n-1}{2} \tag{3.16}$$

where u_{2k} is the coefficient of the tree level superpotential (3.9). If the tree level superpotential is even function, then $f(x)$ is also even function. This means that the heperelliptic curve obtained from this matrix model has Z_2 identification and equivalent to the geometry (3.2) obtained in the reduction of Calabi-Yau geometry. For this symmetric filling we have only to consider $x > 0$. Filing numbers are described as follows,

$$S_0 = \frac{1}{4\pi i} \int_{A_0} y(x) dx \,, \qquad S_i = \frac{1}{2\pi i} \int_{A_i} y(x) \,, \quad i > 0 \,. \tag{3.17}$$

We propose that integrals by symplectic dual periods B_i are described in terms of free energy for the matrix model as follow,

$$\frac{\partial \mathcal{F}_0}{\partial S_i} = \int_{B_i} y(x) dx \,. \tag{3.18}$$

Remaining discussion of this subsection we test these relation explicitly for the special case.

We will consider the matrix models with the simplest potential

$$W_{\text{tree}} = \frac{1}{2} m \Phi^2 + \frac{1}{4} g \Phi^4 \,. \tag{3.19}$$

As in U(N) case we consider one cut solution for this model. Eigenvalues are all placed at $\Phi = 0$ classically.[3] We can calculate this genus zero free energy in the same way as in the case of U(N) gauge theory along the derivation in [23]. Since the eigenvalues λ_i

[3] In our case, there is only one period $S = S_0$ around the origin and another period S_1 vanishes.

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of Φ become continuous in the large-N limit, let us introduce a function $\lambda(x)$ such that $\lambda_k = \lambda(k/N)$ for $\mathrm{SO}(N)$ and $\lambda_k = \lambda(2k/N)$ for $\mathrm{Sp}(N)$. For real symmertic matric model and self-dual quaternionic matrix model, the free energies become same and they are given by the integrals

$$\mathcal{F}_0 = -\frac{g_s^2 N^2}{g_s} \int_0^1 dx \left[\frac{m}{2N}\lambda^2(x) + \frac{g}{4N}\lambda^4(x)\right] + \frac{g_s^2 N^2}{2} \int_0^1 \int_0^1 dx dy \log|\lambda(x) - \lambda(y)| \,. \quad (3.20)$$

We introduce the density of eigenvalues $\rho(\lambda)$ defined as

$$\frac{dx}{d\lambda} = \rho(\lambda), \qquad \int_{-2a}^{2a} d\lambda\, \rho(\lambda) = 1. \quad (3.21)$$

As mentioned above, the geometry derived the matrix models with even function are symmetric. Then we put the interval as symmetric. The equation of motion is rewritten in the large-N limit as

$$\frac{m}{g_s N}\lambda + \frac{g}{g_s N}\lambda^3 = \fint_{-2a}^{2a} d\mu \frac{\rho(\mu)}{\lambda - \mu} \quad (3.22)$$

where the integral means the principal part. The solution is obtained by introducing the new analytic function,

$$\omega(\lambda) = \int_{-2a}^{2a} d\mu \frac{\rho(\mu)}{\lambda - \mu}\,, \quad (3.23)$$

defined for complex λ outside the real interval $(-2a, 2a)$. It behaves as $1/\lambda$ when $|\lambda|$ goes to infinity. When λ approaches the interval $(-2a, 2a)$ it behaves as

$$\omega(\lambda \pm i0) = \frac{m}{g_s N}\lambda + \frac{g}{g_s N}\lambda^3 \mp i\pi\rho(\lambda)\,. \quad (3.24)$$

There is a unique function, which satisfies these requirements

$$\omega(\lambda) = \frac{m}{g_s N}\lambda + \frac{g}{g_s N}\lambda^3 - \left(\frac{m}{g_s N} + 2\frac{g}{g_s N}a^2 + \frac{g}{g_s N}\lambda^2\right)\sqrt{\lambda^2 - 4a^2} \quad (3.25)$$

with the constraint

$$6ga^4 + 2ma^2 = g_s N\,. \quad (3.26)$$

Since only the singular part contributes to the density of eigenvalues, we get explicit function for the density

$$\rho(\lambda) = \frac{1}{2\pi i}\left(\omega(\lambda + i0) - \omega(\lambda - i0)\right) = \frac{1}{\pi}\left(\frac{m}{g_s N} + 2\frac{g}{g_s N}a^2 + \frac{g}{g_s N}\lambda^2\right)\sqrt{\lambda^2 - 4a^2}\,. \quad (3.27)$$

In terms of these quantities we obtain

$$\begin{aligned} \mathcal{F}_0 &= -\frac{g_s^2 N^2}{g_s} \int_{-2a}^{2a} d\lambda\, \rho(\lambda) \left[\frac{m}{2N}\lambda^2(x) + \frac{g}{4N}\lambda^4(x)\right] + \\ &\quad + \frac{g_s^2 N^2}{2} \int_{-2a}^{2a}\int_{-2a}^{2a} d\lambda\, d\mu\, \rho(\lambda)\rho(\mu) \log|\lambda - \mu| \\ &= -\frac{S^2}{48}(b^2 - 1)(9 - b^2) + \frac{S^2}{4}\log b^2\,. \end{aligned} \quad (3.28)$$

where $b^2 = \frac{2m}{g_s N} a^2$. This is the exact result of genus zero free energy. Using the relation (3.26) we get the genus zero free energy [23]

$$\mathcal{F}_0(g_s) = -\frac{(g_s N)^2}{2} \sum_{k=1}^{\infty} \left(-\frac{3 g g_s N}{2m^2} \right)^k \frac{(2k-1)!}{k!(k+2)!} \,. \tag{3.29}$$

For the later discussion for Dijkgraaf-Vafa's conjecture we also calculate the derivation of the genus zero free energy with respect to 't Hooft coupling $S \equiv g_s N$,

$$\frac{\partial \mathcal{F}_0}{\partial S} = -\frac{1}{24} \left[S \left(b^2 - 1\right) \left(9 - b^2\right) + S^2 \left(\frac{\partial b^2}{\partial S} \right) \frac{(b^2-3)(2-b^2)}{b^2} \right] + \frac{S}{2} \log b^2 \,. \tag{3.30}$$

Using the derivative of constraint (3.26) with respect to S

$$\frac{\partial b^2}{\partial S} = -\frac{3 g b^4}{2m^2 + 6 g S b^2} \,, \tag{3.31}$$

and substituting the constraint (3.26) we obtain

$$\frac{\partial \mathcal{F}_0}{\partial S} = -\frac{S}{4}(b^2 - 1) + \frac{S}{2} \log b^2 \,. \tag{3.32}$$

Since $\mu^2 = a^2 = \frac{S}{2g\Delta} b^2$, this result agree with the dual period integral Π (3.6) in the previous subsection.

3.3 The check of Dijkgraaf-Vafa conjecture for $\mathrm{SO}(N)/\mathrm{Sp}(N)$ case

In [18] the Dijkgraaf-Vafa's conjecture is generalized to the unoriented case. With the result (3.29) we can also write down the effective superpotential for $\mathrm{SO}(N)/\mathrm{Sp}(N)$ gauge group [4]

$$\begin{aligned} W_{\text{eff}}(S) &= \hat{N} S \log \left(\frac{S}{\Lambda_0^3} \right) - 2\pi i \tau S + \hat{N} \frac{\partial \mathcal{F}_0}{\partial S} \\ &= \hat{N} S \log \left(\frac{S}{\Lambda_0^3} \right) - 2\pi i \tau S + \hat{N} \left[-\frac{S}{4} \left(b^2 - 1\right) + \frac{S}{2} \log b^2 \right] \end{aligned} \tag{3.33}$$

where $\hat{N} = N - 2$ for $\mathrm{SO}(N)$ and where $\hat{N} = N + 2$ for $\mathrm{Sp}(N)$. This effective superpotential agree exactly with the result (3.7) from the Calabi-Yau manifold with fluxes.

$$\begin{aligned} W_{\text{eff}}(S) &= \hat{N} S \log \left(\frac{S}{\Lambda_0^3} \right) - 2\pi i \tau S - \hat{N} \sum_{k=1}^{\infty} \left(-\frac{3g}{2m^2} \right)^k \frac{S^{k+1}}{2} \frac{(2k-1)!}{k!(k+1)!} \\ &= \cdots + \hat{N} \left[\frac{3}{2} \left(\frac{g}{4m^2} \right) S^2 - \frac{9}{2} \left(\frac{g^2}{8m^4} \right) S^3 + \frac{45}{2} \left(\frac{g^3}{16m^6} \right) S^4 + \cdots \right] , \end{aligned} \tag{3.34}$$

This is the exact effective superpotential summed up all instanton corrections for the $\mathrm{SO}(N)/\mathrm{Sp}(N)$ gauge theories. As already mensioned this model is captured by Φ^4-theory with single line in which non-planar d iagrams are omitted. The diagrams in figure 1 give the contribution to the second term of (3.8).

This result also agree perfectly with (3.8). Thus Dijkgraaf-Vafa's proposal for the unoriented case is explicitly checked for the perturbation around $\Phi = 0$.

[4] In [18] the glueball superfield was identified with $g_s(N \mp 2)$. Since we have considered the large-N limit, we can ignored the ∓ 2. Then we identified the glueball superfield with $g_s N$.

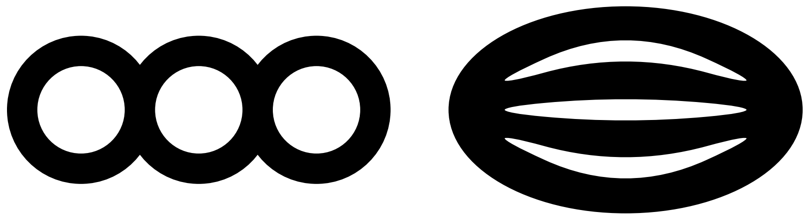

Figure 1: 3-loop diagrams contributing to the fourth order of S.

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